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THE TOPOLOGICAL UNIQUENESS OF TRIPLY PERIODIC MINIMAL SURFACES IN R^3

CHARLES FROHMAN

0. Introduction

The study of topological uniqueness questions related to minimal surfaces was initiated by Lawson in [7]. Lawson proved that if F_1 and F_2 are embedded minimal surfaces in the three-sphere having the same genus, then there is a homeomorphism of the three-sphere taking F_1 to F_2. The method of proof is to show that an embedded minimal surface in a three-manifold of positive Ricci curvature is a Heegaard splitting, and then appeal to a theorem of Waldhausen stating that any two Heegaard splittings of the three-sphere having the same genus are topologically equivalent.

The study of the topological uniqueness of minimal surfaces was furthered by Meeks; a good source for this material is his IMPA lecture notes [8]. Meeks proved that if F is an embedded minimal surface in a closed flat three-manifold, then F is either totally geodesic or a Heegaard splitting. Using this fact he proved that if F_1 and F_2 are minimal surfaces having one boundary component and the same genus in a flat three-ball with convex boundary, then the surfaces are topologically equivalent. Once again the proof is by appeal to Waldhausen's theorem.

In [5] the author showed that any two genus-three minimal surfaces in a flat three-torus are topologically equivalent. The proof is a topological analysis of genus-three Heegaard splittings of T^3 that can be minimal surfaces. In [6] it is shown that genus-three Heegaard splittings of the three-torus are topologically unique. The method of proof is to obtain a minimax surface from the isotopy class of the Heegaard splitting using a technique of Pitts and Rubinstein [9]. The topological uniqueness of genus-three Heegaard splittings can be determined by analyzing the resulting minimal surface.

At the end of [8], Meeks gives a list of fifty open problems in minimal surface theory. Many of them pertain to the topology of minimal surfaces.

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In this paper we give a proof of the first conjecture in the list. A triply periodic minimal surface is a connected properly embedded minimal surface that is invariant under the action of a cocompact lattice in R^3.

Theorem 2.3. *Let F and F' be triply periodic minimal surfaces in a Euclidean 3-space R^3. Then there is a homeomorphism of $h: R^3 \to R^3$ so that $h(F) = F'$.*

The proof we give of this theorem is topological and based on the fact that F and F' cover Heegaard splittings of the three-torus. We use a criterion for the reducibility of Heegaard splittings which was conjectured by R. Craggs [4], and an observation of Meeks. In the first section we will discuss Heegaard splittings and prove Craggs reducibility criterion. In the second section we will prove the following theorem which clearly implies Meeks conjecture.

Theorem 2.2. *Let M be a closed three-manifold which has residually finite fundamental group and is covered by R^3. Let F_1 and F_2 be Heegaard splittings of M, and let $\tilde{F}_1$ and $\tilde{F}_2$ be their inverse images in R^3. Then there is a homeomorphism $h: R^3 \to R^3$, such that $h(\tilde{F}_1) = \tilde{F}_2$.*

In light of this the following conjecture seems reasonable.

Conjecture. Up to topology there is only one properly embedded minimal surface in R^3 having a single end and infinite genus.

It should be mentioned that Boileau and Otal [1] have recently given an alternative solution to Meeks' conjecture. In their paper they prove the topological uniqueness of Heegaard splittings of the three-torus, which implies that the conjecture is true. The author would like to thank Bob Edwards, Chuck Livingston, Bill Meeks and Peter Shalen for helpful conversations on this topic.

1. Heegaard splittings of three-manifolds

In this paper we will only work with orientable three-manifolds. By regular neighborhood we mean closed regular neighborhood. A three-manifold M is said to be irreducible if every sphere in M bounds a ball. All embeddings will be locally flat.

A handlebody B is a three-manifold which is the result of adding 1-handles to a three-sphere. Another way of saying this is that there is a family of properly embedded disks in B so that the result of cutting B along the disks is a ball. The boundary of a handlebody is a closed orientable surface. The genus of a handlebody is the genus of its boundary. Two handlebodies are homeomorphic if and only if they have the same genus. It is easy to see that if X is a finite graph embedded in a three-manifold, then any closed regular neighborhood of X is a handlebody. Conversely if B is a handlebody which is embedded in a three-manifold, we can find a finite graph X that has B as a regular neighborhood. In this case we say that X carries B.

A meridian disk for a handlebody B is a properly embedded disk in B, whose boundary is nontrivial in ∂B. Let B be a handlebody, and D_i a family of disjoint meridian disks. Let H be a regular neighborhood of the union of ∂B and the disks D_i. If no boundary component of H is a sphere, then H is an irreducible three-manifold. We call H a hollow handlebody, and $\partial B \subset H$ the distinguished boundary component of H.

Let M be a closed three-manifold. A Heegaard splitting of M is a surface F such that there are handlebodies B_1 and B_2 in M so that $B_1 \cup B_2 = M$ and $B_1 \cap B_2 = F$. If M is a compact manifold, we say that a surface F contained in M is a hollow Heegaard splitting if there exist H_1 and H_2 embedded in M, which are handlebodies or hollow handlebodies such that $H_1 \cup H_2 = M$, $H_1 \cap H_2 = F$, and F is the distinguished boundary component of the H_i which are hollow handlebodies.

Let F_1 be a genus-one surface with one boundary component which is properly embedded in a three-ball E so that F_1 cuts E into genus-one handlebodies B_1 and B_2 (see Figure 1). There is an operation on Heegaard splittings called stabilization. If $F \subset M$ is a Heegaard splitting, find a ball embedded in M which intersects F in a disk. Remove the part of F lying inside the ball and replace it with F_1. This operation is well defined up to homeomorphism and yields a new Heegaard splitting. We say that a Heegaard splitting is reducible if it is the result of stabilizing some other Heegaard splitting.

The Reidemeister-Singer theorem states that if F and F' are Heegaard splittings of M, then after stabilizing each several times we can obtain new Heegaard splittings which are topologically equivalent [10]. It is a theorem of Waldhausen [11] that any Heegaard splitting of the three-sphere is topologically equivalent to the result of stabilizing the standard two-sphere in the three-sphere the appropriate number of times. From this one can conclude that if F is a surface with one boundary component that is properly embedded in the three-ball and splits the ball into two handlebodies, then F is topologically equivalent to the result of stabilizing a properly embedded disk.

The final classical result which we will be using is Haken's Lemma as generalized by Bonahon and Otal [2], [3]. Let $F \subset M$ be a hollow Heegaard splitting, and suppose that there is a sphere in M which does not bound

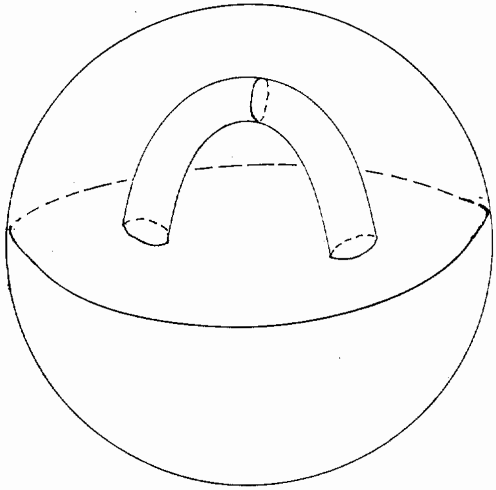

FIGURE 1

a ball. Then there is a sphere in M which does not bound a ball and intersects F in a single simple closed curve.

The following criterion for reducibility was conjectured by R. Craggs [4].

Lemma 1.1. *Let F be a Heegaard splitting of the closed irreducible three-manifold M, and X a graph that carries B. Suppose there is a sphere bounding a ball in M so that some cycle of X lies in the interior of the ball. Then F is reducible.*

Proof. We can assume that M is not the three-sphere, for in that case the theorem is a trivial corollary of Waldhausen's theorem. Now suppose that S is a sphere in M bounding a ball E, and some cycle C of X lies in the interior of E. Remove the interior of a regular neighborhood of C from M, which is small enough to lie inside the same handlebody as C and to lie inside E. Call the manifold obtained $\tilde{M}$. Notice that F is a hollow Heegaard splitting of $\tilde{M}$, and further that S does not bound a ball in $\tilde{M}$ since M is not the three-sphere. By Haken's Lemma, there exists a sphere S' which does not bound a ball in $\tilde{M}$ and intersects F in a single simple closed curve. The sphere S' bounds a ball E' in M. The cycle C lies in E', otherwise S' would bound a ball in $\tilde{M}$. This means that the part of F lying in E' must have genus greater than zero. By Waldhausen's theorem F is reducible.

Lemma 1.2. *Let M be an irreducible three-manifold with residually finite fundamental group whose universal cover is homeomorphic to R^3, and* *let F be a Heegaard splitting of M. Then there exists a cover $p: \tilde{M} \to M$ having finite degree, so that $\tilde{F} = p^{-1}(F)$ is a reducible Heegaard splitting.*

PROOF. Let X be a graph embedded in M which carries F. Let $\tilde{X}$ be the graph in R^3 lying over X in the universal cover of M. Notice that the fundamental group of M acts freely on $\tilde{X}$ without fixed points. If $\tilde{X}$ was a tree, then the fundamental group of M would be free. This would violate the fact that $H^3(M) = Z$. Hence there must be a cycle C in $\tilde{X}$. This cycle is compact, so there exists a ball B in R^3 containing C in its interior. Since the fundamental group of M is residually finite, it is possible to find a subgroup of finite index G so that every deck transformation in G moves B completely off itself. Let $\tilde{M} = R^3/G$. The cycle C embeds in $\tilde{M}$ and is contained in the image of B in $\tilde{M}$ which is a ball. By Lemma 1.1 $\tilde{F}$ is a reducible Heegaard splitting of $\tilde{M}$.

2. Triply periodic minimal surfaces in R^3

Our goal in this section is to prove the theorems stated in the introduction. The next lemma is essentially due to Bill Meeks.

Lemma 2.1. *Suppose that M has R^3 as its universal cover, F is a reducible Heegaard splitting of M, and F' is the result of stabilizing F. Let $p: R^3 \to M$ be the universal covering. Let $\tilde{F}$ and $\tilde{F}'$ be the inverse images of F and F' under p. There is a homeomorphism $h: R^3 \to R^3$ such that $h(\tilde{F}) = \tilde{F}'$.*

Proof. The fundamental group of M can be seen as acting freely and properly discontinuously on R^3 with compact quotient. Hence we can conclude that $\pi_1(M)$ has one end. Since F carries the fundamental group of M, $\tilde{F}$ has one end. Hence we can find an ascending family of compact sets K_i which have the property that $\bigcup K_i = R^3$ and $(\tilde{F} - K_i)$ is connected for all i.

We can find a fundamental domain Δ for the action of $\pi_1(M)$ on R^3 by lifting a spine for M. Number the translates Δ_i of Δ under $\pi_1(M)$, letting $\Delta_1 = \Delta$. Since F is reducible, we can arrange that there is a ball E_1 in Δ_1 so that ∂E_1 intersects F in a single simple closed curve, and $E \cap \tilde{F}$ has genus one. In Figure 2 we have drawn a fundamental domain for the action of $\pi_1(T^3)$ on R^3. Inside we have sketched $\tilde{F} \cap \Delta_1$ where $\tilde{F}$ is a genus-four Heegaard splitting of T^3. The ball in the figure represents E_1. Number the translates of E_1 so that $E_i \subset \Delta_i$.

Choose an arc in $\tilde{F}$ that runs from ∂E_2 into Δ_1. Use the arc to construct an isotopy which moves $\tilde{F} \cap E_2$ into Δ_1 so that what lies inside Δ_1 covers

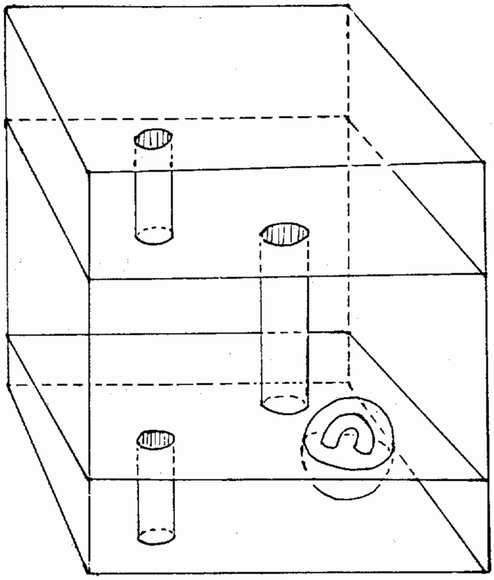

FIGURE 2

the result of stabilizing F once, and the isotopy only moves points which are near the arc. Now choose arcs in $\tilde{F}$ running from ∂E_3 and ∂E_4 into Δ_2, and isotope the parts of $\tilde{F}$ lying inside E_3 and E_4 into Δ_2 so that the end result does not change $\Delta_1 \cap \tilde{F}$, and so that what lies inside Δ_2 is the translate of what now lies inside Δ_1. Now continue dragging handles from E_{2j-1} and E_{2j} into Δ_j so that $\Delta_1, \cdots, \Delta_{j-1}$ are the same after the isotopy, and the part lying inside Δ_j is the translate of what lies in the earlier translates. Furthermore since $\tilde{F} - K_i$ is connected, if Δ_j, E_{2j} and E_{2j-1} are outside of K_i, then we can require that our isotopy does not move points inside K_i. Consequently this process converges pointwise to a homeomorphism $h: R^3 \to R^3$. From our construction $h(\tilde{F}) = \tilde{F}'$.

Proof of Theorem 2.2. Let M be as in the hypothesis, and F_1 and F_2 be two Heegaard splittings of M. Using Lemma 1.2 we can pass to a finite cover M^1 of M so that the inverse images F_1^1 and F_2^1 are reducible Heegaard splittings of M^1. By the Reidemeister-Singer theorem we can stabilize F_1^1 and F_2^1 several times apiece to obtain topologically equivalent Heegaard splittings F_1^S and F_2^S. Hence there is a homeomorphism $g: M^1 \to M^1$ so that $g(F_1^S) = F_2^S$). We will denote the inverse images of F_1^S and F_2^S in the universal cover by using a tilda. Let $\tilde{g}: R^3 \to R^3$ be a lift of g. By Lemma 2.1 there are homeomorphisms h_1, h_2: $R^3 \to R^3$ so that $h_1(\tilde{F}_1) = \tilde{F}_1^S$ and $h_2(\tilde{F}_2) = \tilde{F}_2^S$. The homeomorphism h_1 followed by $\tilde{g}$ followed by h_2^{-1} takes $\tilde{F}_1$ to $\tilde{F}_2$.

Proof of Theorem 2.3. Let $\tilde{F}_1$ and $\tilde{F}_2$ be triply periodic minimal surfaces in R^3. Since the quotient of R^3 by the action of cocompact lattice is homeomorphic to $T^3 = S^1 \times S^1 \times S^1$, we can view the images of $\tilde{F}_1$ and $\tilde{F}_2$ in their respective quotient spaces as being two surfaces F_1 and F_2 lying in T^3. It is a theorem of Meeks [8] that a minimal surface in a closed flat three-manifold is either totally geodesic, or a Heegaard splitting. Since $\tilde{F}_1$ and $\tilde{F}_2$ are connected, F_1 and F_2 are Heegaard splittings of T^3. The hypotheses of Theorem 2.2 are satisfied.

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State University of New York, Buffalo